1 Signals and Spectra

Modulation

- Allows small antenna
- Can match with channel properties (e.g. optical fibre)
- Mixer/Filter/Amplifiers are typically better for high frequencies
- $0.01 < \frac{B}{f_c} < 0.1$ in a practical system
- A large bandwidth requires a large carrier frequency

Main Lobe BW of signal duration T_s

$$\frac{1}{T_s}$$

$$\text{Duration} \times \text{Bandwidth} \geq \frac{1}{4\pi}\text{RMS}$$

Real Signal

$$|M(-f)| = |M(f)|$$
$$\angle M(-f) = -\angle M(f)$$

Fourier Series

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$
$$\forall t \in [-0.5T_0, 0.5T_0]]$$
$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)e^{-j2\pi n f_0 t}dt \equiv c(nf_0)$$

Fundamental Frequency $f_0 = \frac{1}{T_0}$

nth Harmonic $\pm nf_0$ For a real signal:

$$s(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n|\cos(2\pi n f_0 t + \angle c_n)$$
$$\forall t \in [0.5T_0, 0.5T_0]$$

Fourier Transform

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi f t}dt$$

Inverse Fourier Transform

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi f t}df, \quad \forall t \in R$$

3dB bandwidth

$$20\log_{10}\left(\frac{|S(f)|}{|S(f)|_{max}}\right) \leq -3.01(dB)$$

Sinusoid Power

$$\langle|\sin(t)|^2\rangle| = \langle|\cos(t)|^2\rangle = \frac{1}{2}$$

Average Power

$$\langle s^2(t)\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s^2(t)dt, T \to \infty$$

Random Noise

$$S_N(f) = \begin{cases} 0.5N_0 & |f \pm f_c| \leq B/2 \\ 0 & \text{otherwise} \end{cases}$$
$$\left(\frac{S}{N}\right)_b = \frac{P_r}{N_o W}$$

2 Channels

Distortionless Channel

$$y(t) = ks(t - t_0)$$

Envelope/Group Delay

$$\tau_g := -\frac{1}{2\pi}\frac{d\angle H(f_0)}{df}$$

Phase/Carrier Delay

$$\tau_p := -\frac{\angle H(f_c)}{2\pi f_c}$$

Ideal Equaliser

$$H_{eq}(f) = ke^{-j2\pi f t_0}H(f)^{-1}$$

Tapped Delay Filter

$$H_{eq}(f) = d_{M-n}e^{-j2\pi n f\Delta}$$
$$\approx e^{-j2\pi f M\Delta}H(f)^{-1}$$
$$y(t) = \sum_{n=0}^{2M} d_{M-n}x(t - n\Delta)$$
$$\approx s(t - M\Delta)$$
$$\Delta < 0.5/W$$

3 Probability

Bayes' Rule

$$P(X|Y) \equiv \frac{P(X \cap Y)}{P(Y)}$$
$$\equiv \frac{P(Y|X)P(X)}{P(Y)}$$

Expected Value

$$E[X] := \int_{-\infty}^{\infty} x f_x(x)dx$$

Moment order n

$$E[X^n] := \int_{-\infty}^{\infty} x^n f_x(x)dx$$

Variance

$$\sigma_X^2 = E[(X - m_X)^2]$$
$$= E[X^2] - E[X]^2$$

Covariance

$$\text{cov}[X, Y] := E[(X - m_x)(Y - m_y)]$$
$$= E[XY] - E[X]E[Y]$$

Correlation

$$E[XY]$$

Correlation Coefficient

$$\rho_{X,Y} := \frac{\text{cov}[X, Y]}{\sigma_X\sigma_y} \in [-1, 1]$$

Joint pmf

$$F_{X,Y}(x, y) := P(X \leq x, Y \leq y)$$

Joint pdf

$$f_{X,Y}(x, y) := \frac{\delta^2}{\delta_x\delta_y}F_{X,Y}(x, y)$$
$$F_X(x) = F_{X,Y}(x, \infty)$$
$$f_x(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$
$$F_{X,Y}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u, v)dvdu$$

Conditional pdf

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

Mutual Independence

$$f_{Y|X}(y|x) = f_Y(y)$$

Gaussian PDF

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x - m)^2}{2\sigma^2}\right)$$
$$> 0, x \in R$$

$\Phi(x) :=$ cdf of N(0,1)

$Q(x) := 1 - \Phi(x)$, upper tail prob

Jointly Gaussian RVs

Covariance matrix

$$K = E[(X - m)(X - m)']$$
$$f_x \equiv f_{X_1,\ldots,X_n}(x_1, \ldots x_n)$$
$$= \frac{\exp(-0.5(x - m)'K^{-1}(x - m))}{(2\pi)^{n/2}|\det K|^{1/2}}$$

Transformations of RVs

If $Y = g(X)$ is differentiable and 1-1 then:

$$f_Y(y) = f_X(x)/\frac{dy}{dx}$$

If $Z = g(X, Y), W = h(X, Y)$ where the mapping $(g, h) : R^2 \to R^2$ is 1-1 and differentiable then:

$$f_{Z,W}(z, w) = f_{X,Y}(x, y)/|\det J(x, y)|$$
$$J(x, y) := \begin{bmatrix} \frac{\delta z}{\delta x} & \frac{\delta z}{\delta y} \\ \frac{\delta w}{\delta x} & \frac{\delta w}{\delta y} \end{bmatrix}$$

Central Limit Theorem

$$S_n := \sum_{i=1}^{n} X_i$$
$$Z_N := \frac{S_n - n\mu}{\sigma\sqrt{n}}$$
$$\lim_{n\to\infty} P[Z_N \leq z]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp(-x^2/2)dx$$

Characteristic Function

Essentially a Fourier transform

$$\Psi_X(v) := E[e^{jvX}]$$
$$= \int_{-\infty}^{\infty} e^{jvX}f_X(x)dx$$

If X,Y independent

$$\Psi_{X+Y}(v) = \Psi_X(v)\Psi_Y(v)$$
$$E[X^n] = \frac{1}{j^n}\frac{d^n\Psi_X(v)}{dv^n}|_{v=0}$$

3.1 Random Processes

Wide-sense Stationarity

- $m_X(t)$ constant for all time
- $R_X(s, t) \equiv R_X(s - t)$ acf depends only on time difference

Autocorrelation Properties

Average Power $R_X(0) \equiv E[X(t)^2]$

Even $R_X(-\tau) = R_X(\tau)$

Max at 0 $|R_X(\tau)| \leq R_X(0)$

Power Spectral Density (PSD)

$$S_X(f) := \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau$$

Properties:

$$R_X(\tau) := \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}df$$

$E[X(t)^2] := \int_{-\infty}^{\infty} S_X(f)df$ (constant over t)

$$S_X(f) \geq 0$$

For real-valued process

$S_X(f) = S_X(-f)$ White noise

$S_X(f)$ is constant in f

Linear Systems with Stationary Input

$$m_Y = m_X\int_{-\infty}^{\infty} h(\tau)d\tau = m_X H$$
$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\cdot R_X(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2$$
$$S_Y(f) = |H(f)|^2 S_X(f)$$

i.e Output PSD only depends on input PSD and system magnitude response

Cross-correlation

$$R_{XY}(s, t) := E[X(s)Y(t)] \equiv R_{YX}(t, s)$$

Cross-covariance

$$C_{XY}(s, t) := E[(X(s) - m_X(s))\cdot(Y(t) - m_Y(t))]$$
$$\equiv C_{YX}(t, s)$$

Mutually uncorrelated:

$$X_{CY}(s, t) = 0$$

Cross Spectral Density

$$S_{XY}(f) := \mathbf{F}[R_{XY}(\tau)]$$

Bandpass Process

Real WSS random process is

- Lowpass: $S_X(f) = 0\forall|f| > W$, for some $W \geq 0$
- Bandpass: $S_X(f) = 0$ for all f outside $[-(f_0 + W), -(f_0 - W)]\&[f_0 - W, f_0 + W]$, for some $f_0 > W \geq 0$

Bandpass process:

$$X(t) = X_c(t)\cos(2\pi f_0 t) - X_s(t)\sin(2\pi f_0 t)$$

$X_c(t), X_s(t)$ lowpass, jointly WSS

$$X_c(t) = X(t)\cos(2\pi f_0 t) + \hat{X}(t)\sin(2\pi f_0 t)$$
$$X_s(t) = \hat{X}(t)\cos(2\pi f_0 t) - X(t)\sin(2\pi f_0 t)$$

Properties:

$$R_{X_c}(\tau) = R_{X_s}(\tau)$$
$$= R_X(\tau)\cos(2\pi f_0\tau) + \hat{R}_X(\tau)\sin(2\pi f_0\tau)$$
$$S_{X_c}(f) = S_{X_s}(f)$$
$$= S_X(f - f_0) + S_X(f + f_0)$$

for $|f| < W$

4 Trigonometry

$$\cos(x) = \text{Re}(e^{jx}) = \frac{e^{jx} + e^{-jx}}{2}$$
$$\sin(x) = \text{Re}(e^{jx}) = \frac{e^{jx} - e^{-jx}}{2j}$$
$$a\sin(\theta) + b\cos(\theta) = R\sin(\theta + \alpha)$$
$$R = \sqrt{a^2 + b^2} \quad \alpha = \arctan\left(\frac{b}{a}\right)$$
$$\sin(u \pm v) = \sin u\cos v \pm \cos u\sin v$$
$$\cos(u \pm v) = \cos u\cos v \mp \sin u\sin v$$
$$\sin(2u) = 2\sin u\cos u$$
$$\cos(2u) = \cos^2 u - \sin^2 u$$
$$= 2\cos^2 u - 1$$
$$= 1 - 2\sin^2 u$$
$$\sin u\sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u\cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u\cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u\sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

5 Bandpass Signals and Systems

Any Bandpass signal can be alternatively written as

$$s(t) = a(t)\cos\underbrace{(2\pi f_c t + \varphi(t))}_{\theta(t)}$$

$$a(t) = \sqrt{s_c(t)^2 + s_s(t)^2}$$

$$\varphi(t) = \arctan(\frac{s_s(t)}{s_c(t)})$$

Bandpass Narrowband

$f_0 >> B, \pm 0.5B$ of f_0

Bandpass to Lowpass (frequency)

1. Suprress the negative frequencies

$$Z(f) = (1 + \text{sgn}(f))S(f)$$

2. Downshift the spectrum

$$S_L(f) = z(f + f_0)$$

Lowpass to Bandpass (frequency)

1. Upshift

$$Z(f) = S_L(f - f_0)$$

2. Reflect around 0, conjugate, add to PE and scale

$$S(f) = \frac{Z(f) + Z(-f)}{2}$$

Bandpass to Lowpass (time)

1. CE:

$$z(t) \equiv s(t) + j\hat{s}(t)$$

2. Take IFT:

$$S_l(t) = e^{-j2\pi f_0 t}z(t)$$
$$z(t) = e^{j2\pi f_0 t}s_l(t)$$

Lowpass to Bandpass (time)

$$s(t) = \text{Re}[z(t)] = \text{Re}[e^{j2\pi f_0 t}s_l(t)]$$

Bandpass Representation

$$z(t) = A(t)e^{j(2\pi f_c t + \phi(t))}$$
$$s_l = s_c + js_s$$

$$\text{Re}(z(t)) = s(t) = A(t)\cos(2\pi f_c t + \phi(t))$$
$$= \underbrace{s_c(t)}_{\text{In phase}}\cos(2\pi f_0 t) - \underbrace{s_s(t)}_{\text{Quad}}\sin(2\pi f_0 t)$$

$$s_c(t) = s(t)\cos(2\pi f_0 t) + \hat{s}(t)\sin(2\pi f_0 t)$$
$$s_s(t) = -s(t)\sin(2\pi f_0 t) + \hat{s}(t)\cos(2\pi f_0 t)$$

Hilbert Transform

$$H(f) = -j\,\text{sgn}(f)$$
$$h(t) = \frac{1}{\pi t}$$
$$\hat{s} := s * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t - \tau}d\tau$$

Common transform pairs

$$\sin(t) \leftrightarrow -\cos(t)$$
$$\cos(t) \leftrightarrow \sin(t)$$

6 Double Side Band Suppressed Carrier (DSB-SC)

- Difficult to create an in phase local oscillator for demodulation
- Difficult to create an ideal mixer

$$s(t) = m(t)A_c\cos(2\pi f_c t)$$
$$S(f) = 0.5A_cM(f) + 0.25A_cM(f - f_c)$$
$$+0.25A_cM(f + f_c)$$

Quadrature Carrier Multiplexing

Use a complex envelope to contain 2 real LP signals.

$$s(t) = \text{Re}\{z(t)\} = s_c(t)\cos(2\pi f_c t) - s_s(t)\sin(2\pi f_c t)$$
$$s_c(t) = A_c m_1(t)$$
$$s_s(t) = A_c m_2(t)$$

Synchronous Demodulation

Multiply by $\cos(2\pi f_c t + \phi)$ where ϕ represents the phase offset due to an imperfect local oscillator.

$$v(t) = A_c m(t)\cos(2\pi f_c t)^2$$
$$= 0.5A_c m(t)\cos(\phi)$$

If $\phi \approx \pm\pi/2$, *Quadrature Null Effect* causes severe attenuation.

DSB-SC Power

$$P_{DSBSC} = 0.5A_c^2 P_m$$

6.1 DSB-SC Noise Performance

Received signal

$$S(t) = A_c M(t)\cos(2\pi f_c t) + N(t)$$
$$= (A_c M(t) + N_c(t))\cos(2\pi f_c t)$$
$$- N_s(t)\sin(2\pi f_c t)$$
$$S_{dem}(t) = 0.5A_C M(t) + 0.5N_c'(t)$$

N_c' is LPFed in phase noise

$$P_{dem} = 0.25A_c^2 E[M(t)^2] + 0.25E[N_c'(t)^2]$$
$$= 0.25A_c^2 E[M(t)^2] + 0.5N_0 W$$

Signal to noise ratio

$$\left(\frac{S}{N}\right)_o = \left(\frac{S}{N}\right)_b = \frac{A_c^2 P_M}{2N_0 W}$$

7 Amplitude Modulation (AM)

- Uses more power than DSB-SC
- Eliminates phase reversals

$$|m(t)| < \frac{1}{\mu}$$
$$s(t) = A_c(1 + \mu m(t))\cos(2\pi f_c t)$$
$$S(f) = 0.5A_c(\delta(f - f_c) + \mu M(f - f_c))$$
$$+ 0.5A_c(\delta(f + f_c) + \mu M(f + f_c))$$

AM Demodulation

Use a notch filter to remove DC offset. Will be dangerous to perform this on signals with low frequency components due to the notch filter.

AM Power

$$P_{AM} = 0.5A_c^2 + 0.5\mu^2 A_c^2 P_m$$

7.1 AM Noise Performance

Received signal

$$S(t) + A_c(1 + \mu M(t))\cos(2\pi f_c t) + N(t)$$

Demodulated output (envelope detector)

$$Y(t) = \sqrt{[A_c(1 + \mu M(t)) + N_c(t)]^2 + N_s(t)^2]}$$
$$\approx A_c(1 + \mu M(t)) + N_c(t)$$

Notch filter to remove DC

$$Y'(t) \approx \mu A_c M(t) + N_c(t)$$

Signal to noise ratio

$$\left(\frac{S}{N}\right)_o / \left(\frac{S}{N}\right)_b = \frac{\mu^2 P_M}{1 + \mu^2 P_M} < 1$$

Normalised Message

$$\left(\frac{S}{N}\right)_o = \frac{\mu^2 P_{M_N}}{1 + \mu^2 P_{M_N}}\frac{P_r}{N_o W}$$

AM Threshold Effect

- If $N_0 W >> A_c^2$ there is no meaningful SNR.
- Output SNR decreases linearly with power until A_c^2 is around $N_0 W$ at which signal quality suddenly decreases rapidly to zero

8 Single Side-Band (SSB)

- If the channel is LP with BW B Hz then there is no issue
- In a bandpass channel, after modulation the negative side-band uses valuable positive channel bandwidth
- Not suitable for low frequencies

Suppress negative sideband:

$$\tilde{M}(f) := (1 + \text{sgn}(f))M(f) = \begin{cases} 2M(f) & , f > 0 \\ 0 & , f < 0 \end{cases}$$

$$\tilde{m}(t) = m(t) + j\left(\frac{1}{\pi t} * m(t)\right) = m(t) + j\hat{m}(t)$$

Spectrum:

$$S_{SSB}(f) = 0.5A_c\tilde{M}(f - f_c) + 0.5A_c\tilde{M}(-f - f_c)^*$$
$$= \begin{cases} A_c M(f - f_c) & , f > f_c \\ A_c M(f + f_c) & , f < -f_c \\ 0 & \text{elsewhere} \end{cases}$$

Hilbert Implementation(USB)

$$s_{SSB}(t) = A_c m(t)\cos(2\pi f_c t) - A_c\hat{m}(t)\sin(2\pi f_c t)$$
$$= \text{Re}[\tilde{m}(t)A_c e^{j2\pi f_c t}]$$

BPF Implementation (USB)

1. Generate DSB-SC Signal

$$S_{DSBSC}(f) = A_c M(f - f_c) + A_c M(f + f_c)$$

2. Pass through BPF with the passband $[f_c, f_c + B_{BPF}]$

Demodulation using Hilbert Transformer

Difficult to design a Hilbert transformer with sharp phase transition

$$s_c(t) = s(t)\cos(2\pi f_c t) + \hat{s}(t)\sin(2\pi f_c t) = A_c m(t)$$

Demodulation using LPF

Difficult to design a LPF with sharp magnitude transition.

1. Mix with carrier frequency

$$v(t) = s(t)\cos(2\pi f_c t)$$
$$= 0.5A_c m(t)$$
$$+ \underbrace{0.5A_c m(t)\cos(4\pi f_c t) - 0.5A_c\hat{m}(t)\sin(4\pi f_c t)}_{\text{SSB Signal at } 2f_c\text{Hz}}$$

2. LPF with $W < B_{LPF} < 2f_c$

Using a local oscillator out of phase by a moderate ϕ is acceptable for telephony but not music/video with low frequency content

$$m_{rec}(t) = 0.5A_c(\cos(\phi)m(t) + \sin(\phi)\hat{m}(t)$$

$$M_{rec}(f) = 0.5A_c \begin{cases} e^{-j\phi}M(f) & f > 0 \\ e^{j\phi}M(f) & f < 0 \end{cases}$$

SSB Power

$$P_{SSB} = A_c^2 P_m$$

Frequency Division Multiplexing

- Allows multiple messages on the same channel
- Put different bands of SSB modulation on the same carrier
- Crosstalk can be reduced with a guard band and using a low pass filter
- Use a bandpass filter for demodulation

8.1 SSB Noise Performance

Received signal

$$S(t) = A_c M(t)\cos(2\pi f_c t) - A_c \hat{M}(t)\sin(2\pi f_c t) + N(t)$$

Demodulated output

$$Y(t) = 0.5A_c M(t) + 0.5N_c(t)$$

Noise power

$$P_{N_C} = N_0 W$$

Signal to noise ratio

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 P_M}{N_0 W} = \frac{P_R}{N_0 W}$$

9 Vestigial Side Band (VSB)

- Compromise between SSB and DSB-SC
- Partially suppress most of lower (or upper) sidebangs except a vestige
- For demodulation via envelope detection β cannot be too small

$$H_{vsb}(f) = \begin{cases} 2 & f_c + \beta \le f \le f_c + W \\ 0 & 0 \le f \le f_c - \beta \end{cases}$$

for $|w| \le W$

$$H_{vsb}(f_c + w) + H_{vsb}(f_c - w) = 2$$

Cases:

SSB: $2\beta \approx 0$

DSB-SC: $\beta >> W$, with two sidebands and no quadrature component

Demodulation using local oscillator/LPF

1. Mix with local oscillator
2. BPF ($f_m > W$) or will overlap!

$$\begin{aligned} S_{dem}(f) &\propto [S_{VSB}(f + f_c) + S_{VSB}(f - f_c)]H_{LPF}(f) \\ &\propto [H_{VSB}(f + f_c) + H_{VSB}(f - f_c)]M(f) \\ &= 2M(f) \end{aligned}$$

Demodulation using envelope detection

Can be combined with AM: VSB + C

$$s(t) = A_c(1 + \mu m(t))\cos(2\pi f_c t) - \underbrace{A_c \mu\gamma(t)}_{\text{Quad Comp}} \sin(2\pi f_c t)$$

$$\therefore \text{Envelope} \propto A_c\sqrt{(1 + 0.5\mu m(t))^2 + (\mu\gamma(t))^2}$$

For a small μ:

$$\text{Envelope} \propto 1 + \mu m(t)$$

Transition Bandwidth Selection

- When transition bandwidth $2\beta \approx 0$, VSB resembles SSB
- When $\beta >> W$, VSB filter looks like ideal BPF centered at the carrier frequency and hence the signal is approximately a DSB-SC signal.
- For envelope detection β should not too small

10 Angle Modulation

Only the angle contains message information

$$s(t) = A_c\cos(\theta(t)) = A_c\cos(2\pi f_c t + \phi(t))$$

10.1 Phase Modulation (PM)

Phase of carrier is modulated in proportion to message

$$\phi(t) = k_p m(t)$$

$$s_{PM}(t) = A_c\cos(2\pi f_c t + k_p m(t))$$

k_p: *Phase Sensitivity*

Phase modulation index/Maximum Phase Deviation

$$\beta := \max_t|\phi(t)| = k_p \max|m(t)|$$

11 Frequency Modulation (FM)

$$s_{FM}(t) = A_c\cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau)$$

$$m(\tau) = 0, \forall\tau < 0$$

$$\phi(t) = 2\pi k_f \int_0^\tau m(\tau)d\tau$$

Instantaneous Frequency:

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta}{dt} = f_c + k_f m(t)$$

Maximum Frequency Deviation

$$\Delta f := k_f \max_t|m(t)|$$

Single Tone Analysis

$$m(t) = A_m\cos(2\pi W t)$$

Frequency Modulation Index

$$\beta = \frac{\Delta f}{W} = \frac{k_f \max(m(t))}{W}$$

Time domain signal

$$\begin{aligned} s_{FM}(t) &= \text{Re}\{A_c e^{j2\pi k_f \int_0^\tau m(\tau)d\tau + j2\pi f_c t}\} \\ &= A_c\cos(\beta\sin(2\pi W t))\cos(2\pi f_c t) \\ &- A_c\sin(\beta\sin(2\pi W t))\sin(2\pi f_c t) \end{aligned}$$

Narrow-Band FM

In the case where:

$$\beta = \frac{\Delta f}{W} = \frac{k_f A_m}{W} << 1$$

$$\cos x \approx 1 \quad \sin x \approx 1$$

The magnitude spectrum looks like AM, but unlike AM the quadrature carrier is modulated

$$s_{FM}(t) \approx A_c\cos(2\pi f_c t) - 0.5A_c\beta\cos(2\pi(f_c - W)t) + 0.5A_c\beta\cos(2\pi(f_c + W)t)$$

Wide Band Single Tone FM

Unlike in narrow band, β is not small and cannot make assumption

$$\begin{aligned} s_{FM}(t) &= \text{Re}\{A_c e^{j2\pi f_c t + \phi(t))}\} \\ &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nW)t) \end{aligned}$$

Bessel Function

If $\beta << 1$

$$J_0(\beta) \approx 1, J_1(\beta) \approx \beta, J_n(\beta) << \beta$$

FM Power

Independent of message

$$P_{FM} = 0.5A_c^2$$

Carson's Rule

Estimation of the required FM bandwidth, works well for $\beta >> 1$ and $\beta << 1$. Underestimates in range closer to β

$$B_{Carson} = 2(1 + \beta)W = 2(W + \Delta f)$$

Direct FM Generation

- At high frequency, use a *voltage-controller oscillator (VCO).*
- At low frequency, use a parallel resonant circuit by modulating the capacitance of the circuit. At the resonant frequency

Indirect FM Generation

Major disadvantage is drifting carrier frequency

1. Generate narrow-band FM with stable freq using standard linear modulation techniques
2. Frequency multiplication to generate wide-band FM using a non-linear device

Non-linearities in FM

- Strong: when non-linearity is inserted intentionally
- Weak: when non-linearity arises whenever signal levels become too large. FM is impervious to weak non-linearities

11.1 Frequency Demodulation

Deviation ratio

Strength of the demodulated signal increases with deviation ratio.

$$D = \Delta f/W (= \beta \text{ if single tone})$$

FM Power

$$P_{FM} = 0.5A_c^2$$

11.1.1 FM-AM Demodulation

Using differentiator

$$\frac{ds_{FM}(t)}{dt} = A_c(2\pi f_c + 2\pi k_f m(t)) \cdot \sin(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau)$$

$$F\left\{\frac{ds_{FM}(t)}{dt}\right\} = j2\pi f S_{FM}(f)$$

Slope Circuit

The differentiator amplitude response at the carrier frequency is too large so a slope circuit is needed.

$$H(f) = \begin{cases} j2\pi a(f - f_c + C) & |f - f_c| < B/2 \\ H(-f)^* & |f + f_c| < B/2 \\ 0 & \text{otherwise} \end{cases}$$

Output of demodulator:

$$y(t) = -2\pi a A_c(C+k_f m(t))\sin(2\pi[f_c t+k_f\int_0^t m(\tau)d\tau])$$

- $C \geq k_f max_t|m(t)| = \Delta f$ for enveloped detection, common choice is C = B/2
- Demodulated signal amplitude $\propto k_f$

Limiter

- Limits FM signal with constant amplitude
- Used to avoid distorted message due to amplitude distortion in slope circuit of PLL demodulation.

$$v_o(t) = A' \operatorname{sgn}(v_i(t)) = \underbrace{A'c_1\cos(2\pi[f_c t + k_f\int^t m(\tau)d\tau])}_{\text{FM signal with constant amplitude}}$$

11.1.2 Zero Crossings Detector Demodulation

$$f_c + k_f m(t) \approx \frac{\text{no. zero crossings in interval}}{2T}$$

1. Limit FM signal (easier to detect zeros)
2. Apply pulse-detector pulses for each positive/negative transition
3. Continuously integrate over interval T. The integral is proportional to the number of pulses in T.

11.1.3 Feedback Frequency Demodulation

Phase locked loop

VCO with *frequency sensitivity* k_v and *free running frequency* f_v

$$v(t) = -\sin\underbrace{(2\pi f_v t + 2\pi k_v\int_0^t y(\tau)d\tau)}_{=:\theta_v(t)}$$

Multiply received FM signal with VCO output

$$x(t) = -A_c\cos(\theta_c(t))\sin(\theta_v(t)) = 0.5A_c\Big(\underbrace{\sin(\theta_c(t)-\theta_v(t))}_{\text{low difference freq}} - \underbrace{\sin(\theta_c(t)+\theta_v(t))}_{\text{high sum freq}}\Big)$$

1. Pass through LPF to suppress the high frequency term.
2. Amplify phase detector output by a factor K_a
3. Feed back into VCO to keep angle difference between VCO and FM small $\epsilon := \theta_c(t) - \theta_v(t)$

Well designed PLL keeps angle difference small so can approximate linearly

$$\dot\epsilon(t) = 2\pi(f_c - f_v + k_f m(t) - k_v 0.5K_a A_c\sin(\epsilon(t)))$$

$$\therefore y(t) \approx 0.5K_a A_c\epsilon(t) \approx \frac{f_c - f_v + k_f m(t)}{k_v}$$

Ensure $f_c - f_v, \Delta f << 0.5k_v K_a A_c$

11.2 FM Noise Performance

- Output SNR increases with D at cost of increased transmission BW $B = 2(D+1)W$
- Output SNR increases with A_c quadratically, increasing carrier amplitude decreases output noise power
- Dynamic range compression on message increases $\frac{P_M}{\max_t|m(t)|^2}$ improving output SNR

Received signal

$$R(t) = A_c\cos(2\pi f_c(t) + \Phi(t)) + N(t)$$

Noise decomposition

$$N(t) = N_c(t)\cos(2\pi f_c t) - N_s(t)\sin(2\pi f_c t) \equiv V_n(t)\cos(2\pi f_c t + \phi_n(t))$$

After Limiter and BPF

$$S(t) = A\cos(2\pi f_c t + \phi(t) + \angle G(t))$$

$$G(t) := 1 + \frac{V_n(t)}{A_c}e^{j(\phi_n(t)-\phi(t))}$$

Demodulator output

$$Y(t) = 2\pi k_f M(t) + \frac{d\angle G(t)}{dt}$$

$\angle G(t)$ has a nonlinear dependence on message and is difficult to analyse

Large Carrier Approximation

Assume

$$E[V_n(t)^2] = 2N_0 B << A_c^2$$

$$\angle G(t) \approx \frac{N_s(t)\cos(\phi(t)) - N_c(t)\sin(\phi(t))}{A_c}$$

Wideband Approximation

If $W << B/2$ then for all $|\tau| < 1/W$

$$R_{\angle G}(\tau) \approx R_{N_C}(\tau)/A_c^2$$

Output Noise PSD

$$S_Z(f) = \begin{cases} (2\pi f)^2\frac{N_0}{A_c^2} & |f| < W \\ 0 & |f| > W \end{cases}$$

$$P_Z = \frac{8\pi^2 N_0 W^3}{3A_c^2}$$

Signal to noise ratio

$$\left(\frac{S}{N}\right)_o = \frac{3A_c^2 k_f^2 P_M}{2N_0 W^3} = \frac{3D^2 P_M}{\max_t|m(t)|^2}\left(\frac{S}{N}\right)_b$$

Maximum Deviation Ratio

$$20(D+1) < 10^{(S/N)_b/10}$$

Clicks

- If $N_0 B << 0.5A_c^2$ is not satisfied, significant probability that noise envelope $V_n(t) \geq A_c$.
- Small variations in-phase and quadrature noise will occasionally lead to large phase changes of $\pm 2\pi$
- Demodulator differentiates large phase changes yielding impulse like effects heard as clicks

Threshold Effect

To avoid falling below SNR threshold

$$\left(\frac{S}{N}\right)_b > 20(1+D) =: \left(\frac{S}{N}\right)_{b,th}$$

$$\left(\frac{S}{N}\right)_o > \frac{60P_M}{\max_t|m(t)|^2}D^2(1+D) =: \left(\frac{S}{N}\right)_{o,th}$$

FM Parameter Check

$$\left(\frac{S}{N}\right)_o = \frac{3P_M D^2}{\max_t|m(t)|^2}\frac{P_r}{N_0 W} \geq \left(\frac{S}{N}\right)_{o,desired}$$

$$20(D+1) \leq \frac{P_r}{N_0 W}$$

$$2W(D+1) \leq B_{avail}$$

Pre-emphasis/De-emphasis

- High frequencies won't sound as clean due to quadratic FM output noise PSD
- Put message through highpass filter before modulation to emphasis high frequency components
- Pass demodulated signal through lowpass filter to attenuate noise at high frequencies

$$H_d(f) \equiv \frac{1}{1 + jf/f_0}$$

$$H_e(f) = 1/H_d(f) = 1 + jf/f_0$$

New output noise PSD

$$S_{Z,pd}(f) = 4\pi^2\frac{N_0}{A_c^2}\frac{f^2}{1+(f/f_0^2}$$

$$P_{Z,pd} = 8\pi^2\frac{N_0 f_0^3}{A_c^2}\left(\frac{W}{f_0} - \arctan\left(\frac{W}{f_0}\right)\right)$$

Ratio of new SNR to old

$$\left(\frac{S}{N}\right)_{o,pd}/\left(\frac{S}{N}\right)_o = \frac{1}{3}\left(\frac{(W/f_0)^3}{W/f_0 - \arctan(W/f_0)}\right)$$

12 Digital Communications

Pulse Amplitude Modulation

$$s(t) = \sum_{n=-\infty}^{\infty} d_n p(t - nT)$$

d_n Amplitude of pulse

T Sampling period

$p(t)$ Signalling waveform (pulse)

Ideal Sampler

Outputs a modulated sequence of impulses

$$x_{delta}(t) := \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_{delta}(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s})$$

Impulse Train Fourier Series

$$i(t) = \sum_{m=-\infty}^{\infty}\delta(t - mT_s) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T_s}$$

$$c_n \equiv \frac{1}{T_s}$$

12.1 Quantisation

Quantisation Quality

Measured by *mean square error* (MSE)

$$D = E[(X - Q(X))^2]$$

Signal to quantisation noise ratio

$$\text{SQNR} := \frac{E[X^2]}{E[(X - Q(X))^2]}$$

Uniform Quantisation

If X $U[a,b)$, minimise D by partitioning $[a,b)$ into N intervals of equal length

$$N = 2^v$$

$$\Delta := \frac{b-a}{N}$$

$$\hat{x} = \text{Mid-point}$$

$$E[(X - Q(X))^2] = \frac{\Delta^2}{12} = \frac{E[X^2]}{N^2}$$

Optimal SQNR $= N^2$

For nonuniform X, for $N >> 1$ still have

$$E[(X - Q(X))^2] \approx \frac{\Delta^2}{12} = \frac{E[X^2]}{N^2}$$

given $P[X < a \cup X > b]$ is comparatively small

Companding Quantisation

Using a uniform quantiser to generate non-uniform quantisation

$$\hat{x} = g^{-1}(Q_U(g(x)))$$

Where $g(x)$ is a *compressor*, $g^{-1}(x)$ is an *expander*.

μ-Law

$$g(x) = \frac{\ln(1+\mu|x|)}{\ln(1+\mu)}\operatorname{sgn}(x)$$

A-Law

$$g(x) = \begin{cases} \frac{Ax}{1+\ln A} & |x| < 1/A \\ \frac{1+\ln(A|x|)}{1+\ln A}\operatorname{sgn}(x) & |x| \geq 1/A \end{cases}$$

12.2 Waveform Coding

Pulse Code Modulation

1. Pass cont. time signal through anti aliasing LPF
2. Sample and hold at sampling rate $\geq 2W$
3. Quantise each sample
4. Encode each $N \equiv 2^v$ level as v-bit word
5. Convert to serial bit stream, bit rate $r = 1/T_b$
6. Generate regular signalling pulses once every Ts and use bit stream to modulate their amplitudes

Differential PCM

- For a fixed number of quantisation levels N, quantisation noise power increases with variance of signal
- Oversample signal at $f_s > 2W$, successive samples highly correlated and have little variance
- Store last quantised estimate $\hat{X}((k-1)T_s)$ in a delay buffer and quantise $X(kT_s) - \hat{X}((k-1)T_s)$ for higher resolution

At Transmitter

1. At time k find $\hat{Y}_k = Q(X_k - \hat{X}_{k-1})$ where $\hat{X}_{-1} := m_X$
2. Encode $\hat{Y}_k$ into bits and transmit using PCM
3. Update signal estimate $\hat{X}_k = \hat{X}_{k-1} + \hat{Y}_k$

At Receiver

1. Demodulate received PCM signal to get $\hat{Y}_k$
2. Update signal estimate via $\hat{X}_k = \hat{X}_{k-1} + \hat{Y}_k$ where $\hat{X}_{-1} := m_x$

Delta Modulation

- Extreme case of DPCM where a 1-bit quantiser is used $\hat{Y}_k = \pm\Delta$
- Useful when can sample much faster than Nyquist
- Multi level A/D conversion can be slow so this method can be effective instead
- If Δ too large, slow input changes will oscillate
- If Δ too small, delta modulator can't keep pace with fast input changes and gives sustained under/over shoot
- Adaptive ΔM: increase Δ if the last few $\hat{Y}_k$ have had the same sign. E.g.

$$\Delta_k = \Delta_{k-1}(1.5)^{\operatorname{sgn}(\hat{Y}_k\hat{Y}_{k-1})}$$

12.3 Detecting Binary PCM

- Above PCM threshold, extra transmit power does not improve SNR
- SNR gain to bandwidth increase is exponential relationship
- Analog modulation is a better option than high v PCM if enough baseband SNR is available
- Can use error control coding to achieve optimal performance

Assume channel bandwidth sufficient for pulse and only adds uncorrelated white noise

$$R(t) = dp(t) + N(t), 0 \leq t < T_b$$

Noise Limiting Filter

- Pass received signal through non-ideal filter $H(f)$ to limit noise power

$$y(t) = d(h*p)(t) + (h*N)(t), 0 \leq t < T_b$$

- Distortion of pulse from the filter is fine, as long as at the *decision instant* $t' > 0, |(h*p)(t')|$ is large compared to filtered noise $(h*N)(t')$

Optimal Detection Filter

Desired condition for low probability of misdetection

$$|d(h*p)(t')|^2 << \operatorname{var}[(h*N)(t')], d \neq 0$$

Find $h(t)$ to maximise ratio at decision instant

$$\frac{d^2|(h*p)(t')|^2}{E[|h*N)(t')|^2]}$$

Matched Filter

Find $H(f)$ to maximise

$$\eta = \frac{|\int_{-\infty}^{\infty} H(f)P(f)e^{j2\pi ft'}df|^2}{\int_{-\infty}^{\infty}|H(f)|^2 df}$$

$$\eta_{max} = \int_{-\infty}^{\infty}|P(f)|^2 df$$

$$H_{opt}(f) = e^{-j2\pi ft'}P(f)^*$$

$$h_{opt}(t) = p(t'-t)$$

Noise PSD at output

$$E[|(h*N)(t')|^2] = 0.5N_0\int_{-\infty}^{\infty}|H(f)|^2 df$$

Optimal Output Detection SNR

It can be seen the energy of $p(t)$ is important, not its peak value

$$\left(\frac{S}{N}\right) = \frac{2d^2E_p}{N_0}$$

Statistical Properties of Matched Filter Output

Mean

$$E[Y(T_b)|d] = \int_0^{T_b} dp(t)dt = dE_p$$

Variance

$$\operatorname{var}[Y(T_b)|d] = 0.5N_0E_p$$

Probability of Error

If '0' transmitted

$$p_{e0} = Q\Big(\frac{V - d_0E_p}{\sigma_z}\Big)$$

If '1' transmitted

$$p_{e1} = 1 - Q\Big(\frac{V - d_1E_p}{\sigma_z}\Big)$$

Average probability of error

$$p_e = p_0Q\Big(\frac{V-d_0E_p}{\sigma_z}\Big) + p_1\Big(1 - Q\Big(\frac{V-d_1E_p}{\sigma_z}\Big)\Big)$$

Optimal threshold

For equally likely bits $p_0 = p_1 = 0.5$

$$v_{opt} = \frac{(d_0+d_1)E_p}{2}$$

$$p_e = Q\Big((d_1-d_0)\sqrt{\frac{E_p}{2N_0}}\Big)$$

Error Probability vs Average Power

Average *energy per bit*

$$P_rT_b := E_b$$

Unipolar: $P_r = 0.5A^2E_p/T_b$

$$p_e = Q\Big(\sqrt{\frac{P_rT_b}{N_0}}\Big)$$

Polar: $P_r = 0.25A^2E_p/T_b$

$$p_e = Q\Big(\sqrt{\frac{2P_rT_b}{N_0}}\Big)$$

Polar signalling has better performance for given power

Natural Binary Coding

If a natural binary format is used and $(S/N)_b >> v(>> 1)$

Output noise power

$$\approx \underbrace{\frac{|x|^2_{max}}{4^v3}}_{\text{From Quantisation Noise}} + \underbrace{\frac{4|x|^2_{max}p_e}{3}}_{\text{From Channel Noise}}$$

$$\left(\frac{S}{N}\right)_o \approx \frac{3(P_X/|x|^2_{max}}{4^{-v} + 4Q\Big(\sqrt{\frac{(S/N)_b}{v}}\Big)}$$

If $(S/N)_b >> v^2$ quantisation noise dominates

13 Inter-Symbol Interference

Given a distorting channel $H(f)$ which also adds Gaussian white noise

$$\tilde{p}(t) := (p*h*m)(t)$$

$$Z(t) := (h*m*N)(t)$$

Filter output given by

$$Y(t) = \sum_{n=0}^{\infty} d_n\tilde{p}(t-nT_b) + Z(t)$$

If we want to recover d_{k-1} transmitted during previous bit period, the output of the filter is

$$Y(kT_b) = \underbrace{d_{k-1}\tilde{p}(T_b)}_{\text{Desired}} + \underbrace{\sum_{n:n\neq k-1} d_n\tilde{p}((k-n)T_b)}_{\neq 0 \rightarrow \text{ISI}} + \underbrace{Z(kT_b)}_{\text{Noise}}$$

Can mitigate using *Pulse Shaping* or *Equalisation*

Table of Continuous-time Frequency Fourier Transform Pairs

	$f(t) = \mathcal{F}^{-1}\left\{F(f)\right\} = \int_{-\infty}^{+\infty} f(t)e^{j2\pi ft}df$		$F(f) = \mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi ft}dt$	
transform	$f(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f)$	
time reversal	$f(-t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(-f)$	frequency reversal
complex conjugation	$f^*(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F^*(-f)$	reversed conjugation
reversed conjugation	$f^*(-t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F^*(f)$	complex conjugation
	$f(t)$ is purely real	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f) = F^*(-f)$	even/symmetry
	$f(t)$ is purely imaginary	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f) = -F^*(-f)$	odd/antisymmetry
even/symmetry	$f(t) = f^*(-t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f)$ is purely real	
odd/antisymmetry	$f(t) = -f^*(-t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f)$ is purely imaginary	
time shifting	$f(t-t_0)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f)e^{-j2\pi ft_0}$	
	$f(t)e^{j2\pi f_0 t}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f-f_0)$	frequency shifting
time scaling	$f(af)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{\lvert a\rvert}F\left(\frac{f}{a}\right)$	
	$\frac{1}{\lvert a\rvert}f\left(\frac{f}{a}\right)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(af)$	frequency scaling
linearity	$af(t) + bg(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$aF(f) + bG(t)$	
time multiplication	$f(t)g(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f) * G(f)$	frequency convolution
frequency convolution	$f(t) * g(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$F(f)G(f)$	frequency multiplication
delta function	$\delta(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	1	
shifted delta function	$\delta(t-t_0)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$e^{-j2\pi ft_0}$	
	1	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\delta(f)$	delta function
	$e^{j2\pi f_0 t}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\delta(f-f_0)$	shifted delta function
two-sided exponential decay	$e^{-a\lvert t\rvert} \quad a > 0$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{2a}{a^2+4\pi^2 f^2}$	
	$e^{-\pi t^2}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$e^{-\pi f^2}$	
	$e^{j\pi t^2}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$e^{j\pi\left(\frac{1}{4}-f^2\right)}$	
sine	$\sin\left(2\pi f_0 t + \phi\right)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{j}{2}\left[e^{-j\phi}\delta\left(f+f_0\right) - e^{j\phi}\delta\left(f-f_0\right)\right]$	
cosine	$\cos\left(2\pi f_0 t + \phi\right)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{2}\left[e^{-j\phi}\delta\left(f+f_0\right) + e^{j\phi}\delta\left(f-f_0\right)\right]$	
sine modulation	$f(t)\sin\left(2\pi f_0 t\right)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{j}{2}\left[F\left(f+f_0\right) - F\left(f-f_0\right)\right]$	
cosine modulation	$f(t)\cos\left(2\pi f_0 t\right)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{2}\left[F\left(f+f_0\right) + F\left(f-f_0\right)\right]$	
squared sine	$\sin^2(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{4}\left[2\delta(f) - \delta\left(f-\frac{1}{\pi}\right) - \delta\left(f+\frac{1}{\pi}\right)\right]$	
squared cosine	$\cos^2(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{4}\left[2\delta(f) + \delta\left(f-\frac{1}{\pi}\right) + \delta\left(f+\frac{1}{\pi}\right)\right]$	
rectangular	$\operatorname{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & \lvert t\rvert \leqslant \frac{T}{2} \\ 0 & \lvert t\rvert > \frac{T}{2} \end{cases}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$T \operatorname{sinc} Tf$	
triangular	$\operatorname{triang}\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{\lvert t\rvert}{T} & \lvert t\rvert \leqslant T \\ 0 & \lvert t\rvert > T \end{cases}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$T \operatorname{sinc}^2 Tf$	
step	$u(t) = 1_{[0,+\infty]}(t) = \begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{j2\pi f} + \delta(f)$	
signum	$\operatorname{sgn}(t) = \begin{cases} 1 & t \geqslant 0 \\ -1 & t < 0 \end{cases}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{j\pi f}$	
sinc	$\operatorname{sinc}(Bt)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{B}\operatorname{rect}\left(\frac{f}{B}\right) = \frac{1}{B}1_{\left[-\frac{B}{2},+\frac{B}{2}\right]}(f)$	
squared sinc	$\operatorname{sinc}^2(Bt)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{B}\operatorname{triang}\left(\frac{f}{B}\right)$	
n-th time derivative	$\frac{d^n}{dt^n}f(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$(j2\pi f)^n F(f)$	
n-th frequency derivative	$t^n f(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\frac{1}{(-j2\pi)^n}\frac{d^n}{df^n}F(f)$	
	$\frac{1}{1+t^2}$	$\overset{\mathcal{F}}{\Longleftrightarrow}$	$\pi e^{-2\pi\lvert f\rvert}$	